

4 Projectivity of the Moduli of Curves

4.1 Introduction

Recall that $\overline{\mathcal{M}}_g$, the moduli stack of stable curves of genus g , is a proper smooth Deligne–Mumford stack over $\mathrm{Spec}(\mathbb{Z})$, and admits a coarse moduli space $\overline{\mathcal{M}}_g \rightarrow \overline{M}_g$.

Today we will prove that $\overline{M}_g$ is a projective scheme over $\mathrm{Spec}(\mathbb{Z})$. More precisely, we will prove the following:

Theorem 4.1. *Let $\pi: \overline{\mathcal{C}}_g \rightarrow \overline{\mathcal{M}}_g$ be the universal family of stable curves, and $\omega = \omega_{\overline{\mathcal{C}}_g/\overline{\mathcal{M}}_g}$ be the relative dualizing module. Define*

$$\lambda_m := \det \pi_*(\omega^{\otimes m}).$$

Then $\lambda_m^{\otimes d}$ descends to an ample invertible module on $\overline{M}_g$ for some $m > 0$ and $d > 0$.

Remark. $\pi_*(\omega^{\otimes m})$ is locally free of rank g if $m = 1$, of rank $(2m - 1)(g - 1)$ if $m > 1$.

Remark. Most of the ampleness criteria (e.g., Nakai–Moishezon, cohomological) and intersection theory (of invertible modules) works for algebraic spaces.

4.2 Preprocessings

First, descending an invertible module to the coarse space can be done in general:

Theorem 4.2. *Let S be a scheme, and $\mathcal{X}$ be an algebraic stack over S . Assume that $\mathcal{X}$ is quasi-compact, and the inertia $\mathcal{I}_{\mathcal{X}} \rightarrow \mathcal{X}$ is finite. Let $f: \mathcal{X} \rightarrow X$ be the coarse space. Then*

$$f^*: \mathrm{Pic}(X) \rightarrow \mathrm{Pic}(\mathcal{X})$$

is injective, and the cokernel is annihilated by some nonzero integer N .

That is, if λ is an invertible module on $\mathcal{X}$, then there exists an invertible module $\mathcal{L}$ on X such that $\lambda^{\otimes N} \simeq f^*\mathcal{L}$.

Proof. Injectivity follows from that $f_*\mathcal{O}_{\mathcal{X}} = \mathcal{O}_X$.

Using the étale local structure of algebraic stacks with finite inertia (and descent), we reduce to the case that $\mathcal{X} = [U/R]$, where U is affine and $s, t: R \rightrightarrows U$ are finite locally free. Then an invertible module λ on $\mathcal{X}$ is given by an invertible module $\mathcal{L}$ on U together with an isomorphism $\alpha: s^*\mathcal{L} \rightarrow t^*\mathcal{L}$, satisfying descent conditions. Then the transition functions for $\mathrm{Norm}_t(s^*\mathcal{L})$ are R -invariant, so it descends to an invertible module $\mathcal{M}$ on X . But $\mathrm{Norm}_t(s^*\mathcal{L}) \simeq \mathrm{Norm}_t(t^*\mathcal{L}) \simeq \mathcal{L}^{\otimes N}$, where $N = \deg(R/U)$. $\square$

Back to the moduli of curves $f: \overline{\mathcal{M}}_g \rightarrow \overline{M}_g$, fix $d = d(m)$ such that $\lambda_m^{\otimes d} = f^*\mathcal{L}_m$. We need to show that $\mathcal{L}_m$ is relatively ample over $\mathrm{Spec}(\mathbb{Z})$.

Since ampleness is open in a family, and the formation of λ_m commutes with arbitrary base change, we may consider $\overline{\mathcal{M}}_g/k$ where k is an algebraically closed field (e.g., $\overline{\mathbb{F}}_p$). We work over k from now on.

Let $g: X \rightarrow \overline{\mathcal{M}}_g$ be a finite surjective morphism where X is a scheme. Then it suffices to show that $g^*\lambda_m$ is ample, as the composition $f \circ g: X \rightarrow \overline{M}_g$ is also finite.

Theorem 4.3. *Let $\pi: C \rightarrow S$ be a stable family of curves of genus g , where S is proper over k . Assume the moduli map $[\pi]: S \rightarrow \overline{\mathcal{M}}_g$ has finite fibers on k -points. Then*

$$[\pi]^*\lambda_{3d} = \det\left(\pi_*\omega_{C/S}^{\otimes 3d}\right)$$

is ample for $d \gg 0$.

This will be proved using Kollár’s ampleness lemma.

Theorem 4.4 (Kollár's Ampleness Lemma). *Let X be a proper algebraic space over k . Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$ -module of rank n , $d > 0$ be an integer, and*

$$\alpha: \mathrm{Sym}^d(\mathcal{E}) \rightarrow \mathcal{Q}$$

be a quotient such that $\mathcal{Q}$ is locally free of rank q . Write $N = \binom{n+d-1}{d}$. Assume that

- (1) $\mathcal{E}$ is nef; and
- (2) the classifying map $[\alpha]: X \rightarrow [\mathrm{Gr}(N, q)/\mathrm{PGL}_n]$ has finite fibers on k -points.

Then $\det(\mathcal{Q})$ is ample.

Recall that a locally free $\mathcal{O}_X$ -module $\mathcal{E}$ is nef if and only if for every $f: C \rightarrow X$, where C is a smooth projective curve over k , the pullback $f^*\mathcal{E}$ is nef.

Theorem 4.5 (Nefness). *Let $\pi: C \rightarrow S$ be a stable family of curves of genus g , where S is a smooth projective curve over k . Then $\pi_*\omega_{C/S}^{\otimes m}$ is nef for $m \geq 2$.*

Assume nefness and Kollár's ampleness lemma, we can prove Theorem 3.

Proof of Theorem 3. Note that $\omega_{C/S}^{\otimes 3}$ is π -relatively very ample, hence induces a closed immersion

$$\begin{array}{ccc} C & \xrightarrow{i} & \mathbb{P} \\ & \searrow \pi & \downarrow p \\ & & S \end{array}$$

where $\mathbb{P} = \mathbb{P}_S(\pi_*\omega_{C/S}^{\otimes 3})$. Then, for $d \gg 0$ the following hold:

- (1) $\alpha_d: \mathrm{Sym}^d(\pi_*\omega_{C/S}^{\otimes 3}) \rightarrow \pi_*\omega_{C/S}^{\otimes 3d}$ is surjective;
- (2) for every $s \in S(k)$, if $\mathcal{I}_s$ is the ideal of $C_s \hookrightarrow \mathbb{P}_s$, then $\mathcal{I}_s(d)$ is globally generated.

Remark. For (2), it suffices to take $d \geq 6g - 6 = \deg \omega_{C_s/k}^{\otimes 3}$.

We apply the ampleness lemma to α_d . The nefness condition is known.

Note that the formation of α_d commutes with base change. If two points $s, t \in S(k)$ are identified by the classifying map $[\alpha_d]: S \rightarrow [\mathrm{Gr}/\mathrm{PGL}_n]$, then there exists an isomorphism

$$H^0(C_s, \omega_{C_s/k}^{\otimes 3}) \simeq H^0(C_t, \omega_{C_t/k}^{\otimes 3}),$$

that identifies the quotients $\alpha_{d,s}$ and $\alpha_{d,t}$. By (2), $\mathcal{I}_s(d)$ and $\mathcal{I}_t(d)$ are globally generated by $\ker(\alpha_{d,s})$ and $\ker(\alpha_{d,t})$. Thus the isomorphism $\mathbb{P}_s \simeq \mathbb{P}_t$ induces an isomorphism $C_s \simeq C_t$, so s and t are identified by the classifying map $[\pi]: S \rightarrow \overline{\mathcal{M}}_g$. Since $S \rightarrow \overline{\mathcal{M}}_g$ has finite fibers on k -points, so does $S \rightarrow [\mathrm{Gr}/\mathrm{PGL}_n]$. $\square$

4.3 Kollár's Ampleness Lemma

Basic idea: on the Grassmannian $\mathbb{G} = \mathrm{Gr}(N, q)$, we have the universal quotient $\mathcal{O}_{\mathbb{G}}^{\oplus N} \rightarrow \mathcal{Q}$, and $\mathcal{O}_{\mathbb{G}}(1) = \det(\mathcal{Q})$ is very ample; so its finite pullback is ample.

Back to Kollár's ampleness lemma. By the Nakai–Moishezon criterion, we need to show that $\det(\mathcal{Q})$ has positive self-intersection on every integral closed subspace $Z \subset X$. Note that the conditions hold for $\mathcal{E}|_Z$ and $\mathcal{Q}|_Z$, and it suffices to show that $\det(\mathcal{Q}|_Z)$ is big and nef. Also note that being big and nef is preserved under generically finite pullback. Thus it suffices to prove the following.

Theorem 4.6. *Let X be a normal irreducible projective scheme over an algebraically closed field k . Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$ -module of rank n , $d > 0$ be an integer, and*

$$\alpha: \mathrm{Sym}^d(\mathcal{E}) \rightarrow \mathcal{Q}$$

be a quotient such that $\mathcal{Q}$ is locally free of rank q . Write $N = \binom{n+d-1}{d}$ and $\mathbb{G} = \mathrm{Gr}(N, q)$. Assume that

- (1) $\mathcal{E}$ is nef; and
- (2) there exists a dense open $X_0 \subset X$ such that the classifying map $[\alpha]|_{X_0}: X_0 \rightarrow [\mathbb{G}/\mathrm{PGL}_n]$ has finite fibers on k -points.

Then $\det(\mathcal{Q})$ is big and nef.

The classifying map from X to the quotient stack $[\mathbb{G}/\mathrm{PGL}_n]$ is given a PGL_n -torsor on X , namely the frame bundle of $\mathcal{E}$, as follows.

Let $\mathbb{P} = \mathbb{P}_X(\mathcal{H}om(\mathcal{E}, \mathcal{O}_X^{\oplus n}))$, with $p: \mathbb{P} \rightarrow X$. Then there is a tautological map

$$\varphi: \mathcal{O}_{\mathbb{P}}^{\oplus n} = p^* \mathcal{O}_X^{\oplus n} \rightarrow p^* \mathcal{E} \otimes \mathcal{O}_{\mathbb{P}}(1).$$

Let $\mathbb{D} \subset \mathbb{P}$ be the divisor defined by $\det(\varphi) \in H^0(\mathbb{P}, \det(p^* \mathcal{E} \otimes \mathcal{O}_{\mathbb{P}}(1)))$. Let $\mathbb{F} = \mathrm{Fr}(\mathcal{E}) := \mathbb{P} \setminus \mathbb{D}$.

Remark. $\mathbb{F} = \mathrm{Fr}(\mathcal{E})$ represents the following functor: for every X -scheme $f: S \rightarrow X$, $\mathbb{F}(S)$ is the set of isomorphism classes of data $(\mathcal{L}, \psi)$, where $\mathcal{L}$ is an invertible $\mathcal{O}_S$ -module and $\psi: \mathcal{O}_S^{\oplus n} \rightarrow f^* \mathcal{E} \otimes \mathcal{L}$ is an isomorphism.

The action of GL_n on $\mathcal{O}_X^{\oplus n}$ makes $\pi: \mathbb{F} \rightarrow X$ a PGL_n -torsor. We can pullback α to $\mathbb{F}$ to get

$$\tilde{\alpha}: \mathcal{O}_{\mathbb{F}}^{\oplus N} = \mathrm{Sym}^d(\mathcal{O}_{\mathbb{F}}^{\oplus n}) \simeq \mathrm{Sym}^d(\pi^* \mathcal{E} \otimes \mathcal{O}_{\mathbb{F}}(1)) \simeq \pi^* \mathrm{Sym}^d(\mathcal{E}) \otimes \mathcal{O}_{\mathbb{F}}(d) \rightarrow \pi^* \mathcal{Q} \otimes \mathcal{O}_{\mathbb{F}}(d),$$

where $\mathcal{O}_{\mathbb{F}}(1) = \mathcal{O}_{\mathbb{P}}(1)|_{\mathbb{F}}$. Thus we get a Cartesian diagram

$$\begin{array}{ccc} \mathbb{F} & \xrightarrow{[\tilde{\alpha}]} & \mathbb{G} \\ \downarrow \pi & & \downarrow \\ X & \xrightarrow{[\alpha]} & [\mathbb{G}/\mathrm{PGL}_n] \end{array}$$

such that $[\tilde{\alpha}]^* \mathcal{O}_{\mathbb{G}}(1) = \det(\pi^* \mathcal{Q} \otimes \mathcal{O}_{\mathbb{F}}(d)) = \pi^* \det(\mathcal{Q}) \otimes \mathcal{O}_{\mathbb{F}}(qd)$.

Recall that we construct $\mathbb{F}$ as an open subset of $\mathbb{P} = \mathbb{P}_X(\mathcal{H}om(\mathcal{E}, \mathcal{O}_X^{\oplus n})) = \mathbb{P}_X((\mathcal{E}^\vee)^{\oplus n})$. Thus $[\tilde{\alpha}]$ can be viewed as a rational map $[\tilde{\alpha}]: \mathbb{P} \dashrightarrow \mathbb{G}$. Let $b: \mathbb{P}' \rightarrow \mathbb{P}$ be a projective birational morphism that is an isomorphism over $\mathbb{F}$ such that $[\tilde{\alpha}]$ extends to a morphism $f: \mathbb{P}' \rightarrow \mathbb{G}$, and

$$f^* \mathcal{O}_{\mathbb{G}}(1) = b^*(p^* \det(\mathcal{Q}) \otimes \mathcal{O}_{\mathbb{P}}(qd))(-E)$$

for some effective b -exceptional divisor E . Let Z be the scheme-theoretic image of $\mathbb{P}' \rightarrow X \times_{\mathrm{Spec}(k)} \mathbb{G}$.

$$\begin{array}{ccccc} & & \mathbb{P}' & \xrightarrow{f} & \mathbb{G} \\ & \nearrow & \downarrow b & \searrow h & \\ \mathbb{F} & \hookrightarrow & \mathbb{P} & \xrightarrow{g} & \mathbb{G} \\ & \searrow \pi & \downarrow p & \downarrow \rho & \\ & & X & & \end{array}$$

Claim. $g: Z \rightarrow \mathbb{G}$ is generically finite.

Proof. Let $\mathbb{F}_0 = \pi^{-1}(X_0)$, then $h(\mathbb{F}_0)$ is a dense constructible set, so it contains a dense open $Z_0 \subset Z$. Then the fibers of $Z_0(k) \rightarrow \mathbb{G}(k)$ maps injectively to fibers of $X_0(k) \rightarrow \mathbb{G}(k)/\mathrm{PGL}_n(k)$, which are finite, so $g: Z \rightarrow \mathbb{G}$ is generically finite. $\square$

Assume the claim, we know $g^* \mathcal{O}_{\mathbb{G}}(1)$ is big. Let A be an ample divisor on X . Then

$$H^0(Z, g^* \mathcal{O}_{\mathbb{G}}(m)(-\rho^* A)) \neq 0$$

for $m \gg 0$. Pullback to $\mathbb{P}'$, we get

$$\begin{aligned} 0 &\neq H^0(\mathbb{P}', f^* \mathcal{O}_{\mathbb{G}}(m)(-h^* \rho^* A)) \\ &= H^0(\mathbb{P}', b^*(p^* \det(\mathcal{Q})^{\otimes m} \otimes \mathcal{O}_{\mathbb{P}}(mqd))(-mE - b^* p^* A)) \\ &\subset H^0(\mathbb{P}, p^* \det(\mathcal{Q})^{\otimes m} \otimes \mathcal{O}_{\mathbb{P}}(mqd)(-p^* A)) \\ &= H^0(X, \mathrm{Sym}^{mqd}((\mathcal{E}^\vee)^{\oplus n}) \otimes \det(\mathcal{Q})^{\otimes m}(-A)) \end{aligned}$$

by the projection formula, where $b_*\mathcal{O}_{\mathbb{P}'} = \mathcal{O}_{\mathbb{P}}$ since $\mathbb{P}$ is normal, and $p_*\mathcal{O}_{\mathbb{P}}(mqd) = \text{Sym}^{mqd}((\mathcal{E}^\vee)^{\oplus n})$. That is, there is a non-zero map

$$\text{Sym}^{mqd}((\mathcal{E}^\vee)^{\oplus n})^\vee \rightarrow \det(\mathcal{Q})^{\otimes m} \otimes \mathcal{O}_X(-A).$$

Since $\mathcal{E}$ is nef, so is $\text{Sym}^{mqd}((\mathcal{E}^\vee)^{\oplus n})^\vee$ (see [Barton 1971, Tensor Products of Ample Vector Bundles in Characteristic p]). Then the following lemma implies that $\det(\mathcal{Q})$ is big and nef.

Lemma 4.7. *Let X be an integral projective scheme over k . Let $\mathcal{F}$ be a locally free $\mathcal{O}_X$ -module of finite rank, $\mathcal{L}$ be an invertible $\mathcal{O}_X$ -module. Assume the following hold:*

- (a) $\mathcal{F}$ is nef;
- (b) $\mathcal{L}$ is nef;
- (c) there is a non-zero map $\mathcal{F} \rightarrow \mathcal{L}(-A)$ for some ample divisor A .

Then $\mathcal{L}$ is big and nef.

Proof. Exercise. □

4.4 Nefness in Stable Families

Assume k is an algebraically closed field of characteristic $p > 0$. The case in characteristic zero can be obtained by reduction modulo p , and we don't need this case.

Unfortunately we will use theorems in surface theory.

Theorem 4.8 (Ekedahl). *Let X be a smooth minimal surface of general type. Then*

$$h^0(X, \omega_X^{\otimes m}) \leq 1$$

for all $m \geq 2$.

We first consider a special case.

Theorem 4.9. *Let C be a smooth projective curve over k . Let $f: X \rightarrow C$ be a family of nodal curves such that the generic fiber is smooth and of genus $g \geq 2$. Then $f_*\omega_{X/C}^{\otimes m}$ is nef for $m \geq 2$.*

Proof. Suppose $g: C' \rightarrow C$ is a finite cover and $f': X' \rightarrow C'$ is the base change, then

$$g^*f_*\omega_{X/C}^{\otimes m} \simeq f'_*\omega_{X'/C'}^{\otimes m}.$$

So we may assume that C has genus at least 2.

Let $h: X' \rightarrow X$ be a minimal resolution. Since f is a generically smooth family of nodal curves, X has at worst rational double points, so $h_*\omega_{X'/C}^{\otimes m} = \omega_{X/C}^{\otimes m}$, and $f \circ h: X' \rightarrow C$ is also a family of nodal curves. Thus we may assume that X is smooth. Also note that (-1) -curves on X are contained in fibers since we assume C has genus at least 2. Thus we can contract (-1) -curves, and assume X is minimal.

Suppose $f_*\omega_{X/C}^{\otimes m}$ is not nef, then it has an invertible quotient of negative degree by Barton–Kleiman. That is, there is a surjective map

$$f_*\omega_{X/C}^{\otimes m} \rightarrow \mathcal{M}^\vee$$

where $\mathcal{M}$ is an invertible $\mathcal{O}_C$ -module with $\deg(\mathcal{M}) = d > 0$. By pullback along Frobenius of C , we may assume that d is large enough, so that $\mathcal{M} \simeq \mathcal{L} \otimes \omega_C^{\otimes m}$ where $\mathcal{L}$ is very ample. Now we have a surjective map

$$\mathcal{L} \otimes \omega_C^{\otimes m} \otimes f_*\omega_{X/C}^{\otimes m} \twoheadrightarrow \mathcal{O}_C.$$

Let $\mathcal{F} = f^*\mathcal{L} \otimes f^*\omega_C^{\otimes m} \otimes \omega_{X/C}^{\otimes m} = f^*\mathcal{L} \otimes \omega_X^{\otimes m}$, then we have a surjective map $f_*\mathcal{F} \rightarrow \mathcal{O}_C$. Hence

$$h^1(C, f_*\mathcal{F}) \geq h^1(C, \mathcal{O}_C) \geq 2.$$

By the Leray spectral sequence, we get

$$h^1(X, f^* \mathcal{L} \otimes \omega_X^{\otimes m}) = h^1(X, \mathcal{F}) \geq 2.$$

Since $\mathcal{L}$ is very ample, we may assume that $\mathcal{L} \simeq \mathcal{O}_C(D)$ where D is a reduced effective divisor on C such that $F = f^*D$ is a disjoint union of smooth fibers. Note that $\mathcal{O}_X(F)|_F \simeq \mathcal{O}_F$, so $\omega_X^{\otimes m}(F)|_F \simeq \omega_F^{\otimes m}$ by adjunction. Then we have a short exact sequence

$$0 \rightarrow \omega_X^{\otimes m} \rightarrow \omega_X^{\otimes m}(F) \rightarrow \omega_F^{\otimes m} \rightarrow 0$$

Since each component of F has genus $g \geq 2$, we have $H^1(F, \omega_F^{\otimes m}) = 0$ for $m \geq 2$. Thus

$$h^1(X, \omega_X^{\otimes m}) \geq h^1(X, \omega_X^{\otimes m}(F)) \geq 2.$$

Note that X is a smooth minimal surface of general type. This is a contradiction to Ekedahl. $\square$

The following is a generalization to marked case.

Theorem 4.10. *Let C be a smooth projective curve over k . Let $f: X \rightarrow C$ be a family of nodal curves such that the generic fiber is smooth. Let $C_1, \dots, C_n$ be disjoint sections of f that are contained in the smooth locus of f . Then $f_* \omega_{X/C}^{\otimes m}(a_1 C_1 + \dots + a_n C_n)$ is nef for all $m \geq 2$ and $0 \leq a_1, \dots, a_n \leq m$.*

Proof. Prove by induction. Write $D = a_1 C_1 + \dots + a_n C_n$. Assume $a_i < m$. By adjunction we can get

$$0 \rightarrow f_* \omega_{X/C}^{\otimes m}(D) \rightarrow f_* \omega_{X/C}^{\otimes m}(D + C_i) \rightarrow f_* \left(\omega_{X/C}^{\otimes m-a_i-1}|_{C_i} \right) \rightarrow 0.$$

Thus we need

$$(\omega_{X/C} \cdot C_i) = -(C_i^2) \geq 0.$$

If the generic fiber has genus ≥ 2 , this follows from the nefness result, since there is a surjective map

$$f_* \omega_{X/C}^{\otimes m} \twoheadrightarrow f_* \left(\omega_{X/C}^{\otimes m}|_{C_i} \right).$$

If the generic fiber has genus 1, this follows from the canonical bundle formula for elliptic fibrations.

If the generic fiber has genus 0, then $f_* \omega_{X/C}^{\otimes m}(a_1 C_1 + \dots + a_n C_n) = 0$ unless $\sum a_i \geq 2m$. By Hodge index theorem we may assume the C_i appearing in the inductive step satisfies that $(C_i^2) \leq 0$. $\square$

Now we can deduce the generically singular case from the marked generically smooth case.

Let $f: X \rightarrow C$ be a family of nodal curves. Suppose $D \subset X$ is the non-isolated components of the singular locus of f . Let $\nu: X^\nu \rightarrow X$ be the normalization, then

- (1) $f^\nu: X^\nu \rightarrow C$ is a disjoint union of families of nodal curves with smooth generic fibers.
- (2) $\nu^* \omega_{X/C} \simeq \omega_{X^\nu/C}(D^\nu)$, where $D^\nu = \nu^{-1}(D)$.
- (3) D^ν is contained in the smooth locus of f^ν .

By base change, we may assume that D and D^ν are unions of disjoint sections. Then we have a short exact sequence

$$0 \rightarrow f_*^\nu \left(\omega_{X^\nu/C}^{\otimes m}((m-1)D^\nu) \right) \rightarrow f_* \omega_{X/C}^{\otimes m} \rightarrow f_* \mathcal{O}_D \rightarrow 0.$$

The first term is nef, and $f_* \mathcal{O}_D$ is a direct sum of copies of $\mathcal{O}_C$. Thus $f_* \omega_{X/C}^{\otimes m}$ is nef.